

Gravitational energy-momentum in small regions according to Møller's tetrad expression

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Abstract

Møller's tetrad gravitational energy-momentum “tensor” is evaluated for a small vacuum region using an orthonormal frame adapted to Riemann normal coordinates. We find that it does satisfy the highly desired property of being a positive multiple of the Bel-Robinson tensor.

1 Introduction: energy-momentum localization

The localization of energy-momentum for gravitating systems is still an outstanding fundamental problem [1]. The classical attempts to identify a gravitational energy-momentum density for Einstein's *covariant* theory, general relativity, had all led to *non-covariant*, coordinate system dependent expressions, generally referred to as pseudotensors (see, e.g., [2, 3, 4, 5]). As coordinate systems have no physical significance, this led to the idea that there was no physically meaningful gravitational energy-momentum density, and, moreover, that that is just what we should expect from the equivalence principle, (see [6], §20.4).

Meanwhile, in 1961 Møller had constructed an energy-momentum expression which leads to a *tensor* under coordinate transformations [7]. This “tensor” form is achieved, however, by introducing an orthonormal frame, a tetrad (aka vierbein). Thus, while this expression is a tensor with respect to

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coordinate transformations, it depends on the local choice of the orthonormal frame. More precisely, like many other energy-momentum expressions, the value it assigns to a spatial region is *quasi-local* [1]: it depends on the fields only at the boundary of the region. The energy-momentum Møller’s “tensor” expression assigns to a spacetime region thus depends on an object which includes non-physical information, namely the choice of tetrad on the boundary. Nevertheless, largely because of its perceived advantages for energy-momentum localization, Møller’s tetrad expression (which, by the way, also admits an interesting teleparallel representation) has continued to attract interest over the years (see, e.g., [8, 9, 10, 11] and the works cited therein), even though there is no generally accepted frame gauge condition.

In certain special cases, however, there is a natural orthonormal frame; then Møller’s expression yields an unambiguous energy-momentum. In particular this is so asymptotically—at spatial infinity. In that case Møller’s expression (like most others) works well (see [12] for an explicit verification; moreover Møller’s tetrad expression in fact also works well at future null infinity [13]). This asymptotic success is actually not at all surprising; having the proper asymptotic behavior is a relatively weak requirement, for in this weak field region an expression need only have the proper linear theory limit.

The situation is different in the one other situation where there is a natural frame—a case which has, to our knowledge, not been previously investigated for Møller’s expression—namely the small region limit. In this limit, to zeroth order, one should get the material energy-momentum density—a quite weak requirement which follows from the equivalence principle. On the other hand the proposed small *vacuum* region limit is that, to *second order*, one gets a positive multiple of the *Bel-Robinson tensor* [14, 15, 1] (that would be sufficient to guarantee that the energy of a small region was *positive*). Now this latter requirement is especially interesting as a test of proposed energy-momentum densities, since it probes the expression beyond the linear order. It is a strong criterion, capable of excluding many otherwise acceptable expressions, in particular *none* of the classical pseudotensors satisfy this requirement (although certain artificial combinations of them do [15, 16, 17]).

Here, using Riemann normal coordinates and the associated “normal” tetrad, we examine Møller’s expression in the small region limit. We find that Møller’s expression *naturally* satisfies this highly desirable vacuum Bel-Robinson property.

2 Møller's energy-momentum tensor

A gravitational energy-momentum density is easily derived from Einstein's equations expressed in terms of differential forms:

$$R^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu = -2\kappa T_\mu. \quad (1)$$

Here $\kappa = 8\pi G/c^4 = 8\pi G$ is the coupling constant, $R^\alpha{}_\beta$ is the curvature 2-form, $T_\mu = T^\nu{}_\mu \eta_\nu$ is the source energy-momentum 3-form, and we are using Trautman's convenient dual form basis $\eta^{\alpha\cdots} := *(\vartheta^\alpha \wedge \dots)$, where ϑ^α is the co-frame. The left hand side of (1) is just $-2G^\nu{}_\mu \eta_\nu$, the Einstein tensor expressed as a 3-form. (Our conventions, unless otherwise stated, follow MTW [6].) Using the definition of the curvature 2-form in terms of the connection one-form and extracting an exact differential leads to

$$\begin{aligned} R^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu &:= (d\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta) \wedge \eta_\alpha{}^\beta{}_\mu \\ &\equiv d(\Gamma^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu) + \Gamma^\alpha{}_\beta \wedge d\eta_\alpha{}^\beta{}_\mu + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu \\ &\equiv d(\Gamma^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu) + \Gamma^\alpha{}_\beta \wedge \Gamma^\lambda{}_\mu \wedge \eta_\alpha{}^\beta{}_\lambda - \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu \end{aligned} \quad (2)$$

where we have used $D\eta_\alpha{}^\beta{}_\mu = 0$, which follows since the connection is metric compatible and torsion free. Using this expansion we can rewrite the Einstein equation (1) in a neat form (which is remarkably similar to the form used by Einstein when he was still searching for a good gravity theory [18]):

$$dp_\mu = 2\kappa \mathcal{P}_\mu, \quad (3)$$

where the energy-momentum flux 2-form is

$$p_\mu := -\Gamma^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu, \quad (4)$$

and the current is the *total energy-momentum* density (3-form)

$$\mathcal{P}_\mu := t_\mu + T_\mu, \quad (5)$$

which “automatically” satisfies the current conservation relation $d\mathcal{P}_\mu = 0$ [5]. This total energy-momentum current “complex” includes the (non-covariant) gravitational energy-momentum density

$$t_\mu := (2\kappa)^{-1} \left(\Gamma^\alpha{}_\beta \wedge \Gamma^\lambda{}_\mu \wedge \eta_\alpha{}^\beta{}_\lambda - \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu \right). \quad (6)$$

According to this prescription the total energy-momentum within a region is given by

$$P_\mu(V) := \int_V \mathcal{P}_\mu = (2\kappa)^{-1} \oint_{\partial V} p_\mu. \quad (7)$$

The volume integral form would lead one to expect that the value depends on the quantities and choice of frame throughout the region, but the closed 2-surface integral shows that the value is *quasi-local*. The value is still *non-covariant*: it depends on the choice of frame—but, as we have already pointed out, only on the choice at (and, through the connection, near) the boundary.

If the frame is *holonomic* then the flux integrand,

$$p_\mu := -\Gamma^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu \equiv -\Gamma^\alpha{}_{\beta\gamma} g^{\beta\sigma} \delta_{\alpha\sigma\mu}^{\tau\rho\gamma} \frac{1}{2} \eta_{\tau\rho}, \quad (8)$$

is the Freud *superpotential* [19] (written as a 2-form) and the gravitational energy-momentum density is the Einstein *pseudotensor* 3-form. If, on the other hand, one chooses the frame to be orthonormal, then *these same formal expressions* (for an earlier observation of this formal correspondence see [20]) become those proposed by Møller [7] in 1961 (by the way, a differential form construction of these expressions virtually the same as ours was presented some time ago by Wallner [21]); the noteworthy thing is that the tetrad expressions are *tensors*—under coordinate transformations. Although they are completely independent of the choice of coordinates, they do depend on the choice of tetrad; in this important sense they are non-covariant. More specifically the energy-momentum values they determine are quasi-local: they depend on the choice of tetrad, but only on the choice at and near the boundary.

3 Riemann normal coordinates and normal tetrads

To find the energy-momentum within a small region surrounding a particular point, we look to the 3-form \mathcal{P}_μ , expanding it in a power series. For this purpose we choose Riemann normal coordinates x^i centered at the selected point. The Maclauren-Taylor expansion of the holonomic components of the metric and connection are well known (see, e.g. [6], §11.6):

$$g_{ij}|_0 = \bar{g}_{ij}, \quad \partial_k g_{ij}|_0 = 0, \quad 3\partial_{kl} g_{ij}|_0 = -R_{ikjl} - R_{iljk}, \quad (9)$$

$$\Gamma^i_{jk}|_0 = 0, \quad 3\partial_l \Gamma^i_{jk}|_0 = -R^i_{jkl} - R^i_{kjl}. \quad (10)$$

Here $\bar{g}_{ij} = \text{diag}(-+++)$ is the Minkowski metric. In the associated “normal” orthonormal frame, the coframe $\vartheta^\alpha = \vartheta^\alpha_k dx^k$ and connection one-form $\Gamma^\alpha_{\beta k} dx^k$ components take closely related analogous values:

$$\vartheta^\alpha_j|_0 = \delta^\alpha_j, \quad \partial_k \vartheta^\alpha_j|_0 = 0, \quad 6\partial_{kl} \vartheta^\alpha_j|_0 = -R^\alpha_{kjl} - R^\alpha_{ljk}, \quad (11)$$

$$\Gamma^\alpha_{\beta j}|_0 = 0, \quad 2\partial_k \Gamma^\alpha_{\beta l}|_0 = R^\alpha_{\beta kl}. \quad (12)$$

It is readily verified that these values satisfy, to the appropriate order, the two relations which transform the metric and connection coefficients between the holonomic and orthonormal frames:

$$g_{ij} = \bar{g}_{\alpha\beta} \vartheta^\alpha_i \vartheta^\beta_j, \quad \vartheta^\beta_j \Gamma^\alpha_{\beta i} = \Gamma^k_{ji} \vartheta^\alpha_k - \partial_i \vartheta^\alpha_j. \quad (13)$$

4 Small region values

Expanding \mathcal{P}_μ using Riemann normal coordinates and the associated normal tetrad gives to zeroth order, unsurprisingly, only the source energy momentum density—just as it should according to the *equivalence principle*. In vacuum regions \mathcal{P}_μ reduces to t_μ (6), and the leading non-vanishing value appears at the second order:

$$2\kappa \mathcal{P}_\mu = \Gamma^{\alpha\beta} \wedge \Gamma^\lambda_\mu \wedge \eta_{\alpha\beta\lambda} - \Gamma^\alpha_\gamma \wedge \Gamma^{\gamma\beta} \wedge \eta_{\alpha\beta\mu} \quad (14)$$

$$\begin{aligned} &\simeq \frac{x^l x^m}{4} \left(R^{\alpha\beta}_{li} R^\lambda_{\mu mj} - \delta^\lambda_\mu R^\alpha_{\gamma li} R^{\gamma\beta}_{mj} \right) dx^i \wedge dx^j \wedge \eta_{\alpha\beta\lambda} \\ &\simeq \frac{x^l x^m}{4} \left(R^{\alpha\beta}_{l\sigma} R^\lambda_{\mu m\delta} - \delta^\lambda_\mu R^\alpha_{\gamma l\sigma} R^{\gamma\beta}_{m\delta} \right) \delta^{\nu\sigma\delta}_{\alpha\beta\lambda} \eta_\nu \\ &= \frac{x^l x^m}{4} \left(2R_{\mu\lambda m\delta} R^{\nu\lambda}_l{}^\sigma - \frac{1}{2} \delta^\nu_\mu R^{\gamma\sigma\delta}_l R_{\gamma\sigma\delta m} \right) \eta_\nu \end{aligned} \quad (15)$$

$$= \frac{x^l x^m}{4} B^\nu_{\mu lm} \eta_\nu, \quad (16)$$

proportional to the Bel-Robinson tensor:

$$B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R^\lambda_{\beta\nu}{}^\sigma + R_{\alpha\lambda\nu\sigma} R^\lambda_{\beta\mu}{}^\sigma - \frac{1}{2} g_{\alpha\beta} R^{\gamma\sigma\delta}_\mu R_{\gamma\sigma\delta\nu}. \quad (17)$$

In this calculation we have used the vanishing of the Ricci tensor in vacuum and some well known curvature tensor symmetry properties.

Integrating over a small coordinate sphere in the surface $x^0 = 0$, using

$$\int x^l x^m d^3x = \frac{1}{3} \delta^{lm} \int r^2 d^3x, \quad l, m = 1, 2, 3 \quad (18)$$

and the traceless property of the Bel-Robinson tensor gives

$$P_\mu \simeq (2\kappa)^{-1} B^0_{\mu lm} \delta^{lm} \frac{4\pi}{3 \cdot 4 \cdot 5} r^5 = (2\kappa)^{-1} B^0_{\mu 00} \frac{4\pi}{60} r^5 = \frac{1}{240G} B^0_{\mu 00} r^5. \quad (19)$$

This result is best appreciated when expressed in terms of the (traceless, symmetric) electric and magnetic parts of the Weyl tensor, $E_{ab} := R_{0a0b}$, $H_{ab} := \frac{1}{2} \epsilon_{acd} R^c_{d0b}$. We then have a value similar to that in electrodynamics:

$$P^\mu \simeq \frac{r^5}{240G} (E_{ab} E^{ab} + H_{ab} H^{ab}, -2\epsilon^{cab} E_{ad} H^d_{b}); \quad (20)$$

the most important feature is $P^0 \geq |P^i| \geq 0$.

5 Conclusion

Thus the desired Bel-Robinson property is *naturally* satisfied for Møller's energy-momentum density. An important consequence is that the gravitational energy according to this measure is *positive*, at least to this order. (We expected this positivity result since in fact Møller's tetrad expression has an associated positive energy proof [8].)

We stress that the vacuum small region Bel-Robinson property is a strong test capable of excluding many otherwise acceptable expressions; in particular *none* of the classical pseudotensors satisfy this requirement (although certain quite artificial combinations of them do [15, 16, 17]). Once again Møller's 1961 tetrad energy-momentum "tensor" stands out as one of the best descriptions for gravitational energy-momentum.

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